

On McCoy Condition and Semicommutative Rings

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Abstract

Let R be a ring, σ an endomorphism of R , I a right ideal in $S = R[x; \sigma]$ and M_R a right R -module. We give a generalization of McCoy's Theorem [16], by showing that, if $r_S(I)$ is σ -stable or σ -compatible. Then $r_S(I) \neq 0$ implies $r_R(I) \neq 0$. As a consequence, if $R[x; \sigma]$ is semicommutative then R is σ -skew McCoy. Moreover, we show that the Nagata extension $R \oplus_{\sigma} M_R$ is semicommutative right McCoy when R is a commutative domain.

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1 Introduction

Throughout the paper, R will always denote an associative ring with identity and M_R will stand for a right R -module. Given a ring R , the polynomial ring with an indeterminate x over R is denoted by $R[x]$. According to Nielson [18], a ring R is called *right McCoy* (resp., *left McCoy*) if, for any polynomials $f(x), g(x) \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp., $sg(x) = 0$) for some $0 \neq r \in R$ (resp., $0 \neq s \in R$). A ring is called *McCoy* if it is both left and right McCoy. By McCoy [15], commutative rings are McCoy rings. Recall that a ring R is *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is obvious that commutative rings are reversible and reversible rings are semicommutative, but the converse do not hold, respectively. Recently, Nielson [18], proved that reversible rings are McCoy. In [6, Corollary 2.3], it was claimed that all semicommutative rings were McCoy. However, Hirano's claimed assumed that if

R is semicommutative then $R[x]$ is semicommutative, but this was later shown to be false [13, Example 2]. Nielson [18, Theorem 2] shows that reversible rings are McCoy and he gives an example of a semicommutative ring which is not right McCoy. Recall that a ring is *reduced* if it has no nonzero nilpotent elements. A module M_R is called *Armendariz* if whenever polynomials $m = \sum_{i=0}^n m_i x^i \in M[x]$ and $f = \sum_{j=0}^m a_j x^j \in R[x]$ satisfy $mf = 0$, then $m_i a_j = 0$ for each i, j . A ring R is Armendariz, if R_R is Armendariz. Reduced rings are Armendariz and Armendariz rings are McCoy. We have the following diagram:

$$\left. \begin{array}{l} R \text{ is reversible} \\ R[x] \text{ is semicommutative} \\ R \text{ is Armendariz} \end{array} \right\} \Rightarrow R \text{ is McCoy}$$

An Ore extension of a ring R is denoted by $R[x; \sigma, \delta]$, where σ is an endomorphism of R and δ is a σ -derivation, i.e., $\delta: R \rightarrow R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Recall that elements of $R[x; \sigma, \delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in R and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. For $\delta = 0$, we put $R[x; \sigma, 0] = R[x; \sigma]$. Baser et al. [5], introduced σ -skew McCoy for an endomorphism σ of R . A ring R is called σ -skew McCoy, if for any nonzero polynomials $p(x) = \sum_{i=0}^n a_i x^i$ and $q(x) = \sum_{j=0}^m b_j x^j \in R[x; \sigma]$, $p(x)q(x) = 0$ implies $p(x)c = 0$ for some nonzero $c \in R$, and they have proved the following:

$$\left. \begin{array}{l} R[x; \sigma] \text{ is right McCoy} \\ R[x; \sigma] \text{ is reversible} \end{array} \right\} \Rightarrow R \text{ is } \sigma\text{-skew McCoy}$$

Hong et al. [11], proved that if σ is an automorphism of R and I a right ideal of $S = R[x; \sigma, \delta]$ then $r_S(I) \neq 0$ implies $r_R(I) \neq 0$, which is a generalization of McCoy's Theorem [16].

In this paper, we give another generalization of McCoy's Theorem, by showing that for any right ideal I of $S = R[x; \sigma]$, we have $r_S(I) \neq 0$ implies $r_R(I) \neq 0$ when R is σ -compatible or $r_S(I)$ is σ -stable. As a consequence, if $R[x; \sigma]$ is semicommutative then R is σ -skew McCoy. We obtain a generalization of [5, Corollary 6] and [6, Corollary 2.3]. Furthermore, we show some results on Nagata extension. For a commutative ring R , we have

1) If R is a domain, then

(a) M_R is Armendariz if and only if $R \oplus_\sigma M_R$ is Armendariz.

- (b) The ring $R \oplus_\sigma M_R$ is semicommutative and right McCoy.
- 2) If R and M_R are Armendariz such that M_R satisfies the condition (\mathcal{C}_σ^2) , then $R \oplus_\sigma M_R$ is Armendariz.

2 A Generalization of McCoy's Theorem

McCoy [16], proved that for any right ideal I of $S = R[x_1, x_2, \dots, x_n]$ over a ring R , if $r_S(I) \neq 0$ then $r_R(I) \neq 0$. This result was extended by Hong et al. [11] to many skew polynomial rings, where σ is an automorphism of R . Herein, we'll extend McCoy's Theorem to skew polynomial rings of the form $R[x; \sigma]$ with σ an endomorphism of R . According to Annin [2], a ring R is σ -compatible, if for any $a, b \in R$, $ab = 0$ if and only if $a\sigma(b) = 0$. Let σ be an endomorphism of R and I an ideal of R , we say that the ideal I is σ -stable, if $\sigma(I) \subseteq I$. Let σ be an endomorphism of a ring R , then for any $f = \sum_{i=0}^n a_i x^i \in R[x; \sigma]$, we denote the polynomial $\sigma(f)$ by $\sigma(f) = \sum_{i=0}^n \sigma(a_i) x^i$.

Theorem 2.1. *Let R be a ring, σ an endomorphism of R and I a right ideal in $S = R[x; \sigma]$. Suppose that $r_S(I)$ is σ -stable or σ -compatible. If $r_S(I) \neq 0$ then $r_R(I) \neq 0$.*

Proof. Suppose that $r_R(I) \neq 0$. If $I = 0$, then it's trivial. Assume that $I \neq 0$. Let $g(x) = \sum_{j=0}^m b_j x^j \in r_S(I)$, we can set $b_m \neq 0$. If $m = 0$, we're done, so we can suppose that $m \geq 1$. In this situation, if $Ib_m = 0$, then we're done. Otherwise, there exists $0 \neq f(x) = \sum_{i=0}^n a_i x^i \in I$, such that $f(x)b_m \neq 0$.

Case 1. If $r_S(I)$ is σ -stable, then $a_i x^i b_m \neq 0$ for some $i \in \{0, 1, \dots, n\}$, therefore $a_i x^i g(x) \neq 0$. Take $p = \max\{i | a_i x^i g(x) \neq 0\}$. Then $a_p \sigma^p(g(x)) \neq 0$ and $a_i x^i g(x) = 0$ for $i \geq p+1$. We obtain $a_p \sigma^p(b_m) = 0$ from $f(x)g(x) = 0$. Also, we have $I(a_p \sigma^p(g(x))) = (Ia_p) \sigma^p(g(x)) = 0$ because I is a right ideal of S and $\sigma^p(g(x)) \in r_S(I)$. So $0 \neq a_p \sigma^p(g(x)) \in r_S(I)$. We can write $a_p \sigma^p(g(x)) = a_p \sigma^p(b_0) + a_p \sigma^p(b_1)x + \dots + a_p \sigma^p(b_\ell)x^\ell$, where $a_p \sigma^p(b_\ell) \neq 0$ and $\ell < m$. If $\ell = 0$ then $Ia_p \sigma^p(b_\ell) = 0$, so $0 \neq a_p \sigma^p(b_\ell) \in r_R(I)$. Otherwise, $\ell \geq 1$, then we'll consider $a_p \sigma^p(g(x))$ in place of $g(x)$. $0 \neq h(x) = \sum_{k=0}^s c_k x^k \in I$ such that $h(x)a_p \sigma^p(b_\ell) \neq 0$. We can find q as the largest integer such that $c_q \sigma^q(a_p \sigma^p(g(x))) \neq 0$ and then $0 \neq c_q \sigma^q(a_p \sigma^p(g(x))) \in r_S(I)$ such that the degree of $c_q \sigma^q(a_p \sigma^p(g(x)))$ is smaller than one of $a_p \sigma^p(g(x))$.

Case 2. If R is σ -compatible, then $a_i \sigma^i(b_m) \neq 0$ for some $i \in \{0, 1, \dots, n\}$, also $a_i b_m \neq 0$, by σ -compatibility of R . Therefore $a_i g(x) \neq 0$ for some $i \in \{0, 1, \dots, n\}$. Take $p = \max\{i | a_i g(x) \neq 0\}$, so $a_p g(x) \neq 0$ and $a_{p+1}g(x) = \dots = a_n g(x) = 0$, thus $a_i b_j = 0$ for $i \in \{p+1, \dots, n\}$ and $j \in \{0, 1, \dots, m\}$. For $i \in \{p+1, \dots, n\}$, we have

$a_i x^i g(x) = \left(\sum_{j=0}^m a_i \sigma^i(b_j) x^j \right) x^i = 0$, with σ -compatibility and the previous condition. On other, we get $a_p b_m = 0$ from $f(x)g(x) = 0$. So that the degree of $a_p g(x)$ is less than m such that $a_p g(x) \neq 0$. But $I(a_p g(x)) = (I a_p)g(x) = 0$ since I is a right ideal of S , so $0 \neq a_p g(x) \in r_S(I)$. We can write $a_p g(x) = \sum_{k=0}^{\ell} a_p b_k x^k$ with $a_p b_{\ell} \neq 0$ and $\ell < m$. We have the two possibilities: If $\ell = 0$ then $a_p g(x)$ is a nonzero element in $r_R(I)$. Otherwise, $\ell \geq 1$. Then we'll consider $a_p g(x)$ in place of $g(x)$. We have two cases $I(a_p b_{\ell}) = 0$ or $I(a_p b_{\ell}) \neq 0$. The first implies $0 \neq a_p b_{\ell} \in r_R(I)$, for the second, there exists $0 \neq h(x) = \sum_{k=0}^s c_k x^k \in I$ such that $h(x)a_p b_{\ell} \neq 0$. Here, we can find q as a the largest integer such that $c_q a_p g(x) \neq 0$ and then $0 \neq c_q a_p g(x) \in r_S(I)$ such that the degree of $c_q a_p g(x)$ is smaller than one of $a_p g(x)$.

Continuing with the same manner (in the two cases), we can produce elements of the forms $0 \neq a_{t_1} a_{t_2} \cdots a_{t_s} \sigma^{t_1+t_2+\cdots+t_s} g(x)$ (resp., $0 \neq a_{t_1} a_{t_2} \cdots a_{t_s} g(x)$) in $r_S(I)$, with $s \leq m$ and the degree of these polynomials is zero. Thus $a_{t_1} a_{t_2} \cdots a_{t_s} \sigma^{t_1+t_2+\cdots+t_s} g(x) \in r_R(I)$ (resp., $0 \neq a_{t_1} a_{t_2} \cdots a_{t_s} g(x) \in r_R(I)$). Therefore $r_R(I) \neq 0$. \square

Corollary 2.2 ([6, Theorem 2.2]). *Let $f(x)$ be an element of $R[x]$. If $r_{R[x]}(f(x)R[x]) \neq 0$ then $r_{R[x]}(f(x)R[x]) \cap R \neq 0$*

Proof. Consider the right ideal $I = f(x)R[x]$. \square

Corollary 2.3. *Let R be a ring, σ an endomorphism of R and I a right ideal of $S = R[x; \sigma]$. If S is semicommutative, then $r_S(I) \neq 0$ implies $r_R(I) \neq 0$.*

According to Clark [4], a ring R is said to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. Following Zhang and Chen [21], a ring R is said to be σ -semicommutative if, for any $a, b \in R$, $ab = 0$ implies $aR\sigma(b) = 0$. A ring R is called *right (left) σ -reversible* if whenever $ab = 0$ for $a, b \in R$, $b\sigma(a) = 0$ ($\sigma(b)a = 0$). A ring R is called σ -reversible if it is both right and left σ -reversible. Hong et al. [7], proved that, if R is σ -rigid then R is quasi-Baer if and only if $R[x; \sigma]$ is quasi-Baer. Recently, Hong et al. [10], have proved the same result when R is semi-prime and all ideals of R are σ -stable.

Proposition 2.4. *Let R be a σ -semicommutative ring. If $R[x; \sigma]$ is quasi-Baer then R so is.*

Proof. Let I be a right ideal of R . We have $r_{R[x; \sigma]}(IR[x; \sigma]) = eR[x; \sigma]$ for some idempotent $e = e_0 + e_1 x + \cdots + e_m x^m \in R[x; \sigma]$. By [3, Proposition 3.9], $r_R(IR[x; \sigma]) = e_0 R$. Clearly, $r_R(IR[x; \sigma]) \subseteq r_R(I)$. Conversely, let $b \in r_R(I)$ then $Ib = 0$, since R is σ -semicommutative, we have $IR[x; \sigma]b = 0$, so $b \in r_{R[x; \sigma]}(IR[x; \sigma])$. Therefore $r_R(I) = e_0 R$. \square

Example 2.5. Let \mathbb{Z} be the ring of integers and consider the ring

$$R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \pmod{2}\}$$

and $\sigma: R \rightarrow R$ defined by $\sigma(a, b) = (b, a)$.

1) $R[x; \sigma]$ is quasi-Baer and R is not quasi-Baer, by [7, Example 9].

2) R is not σ -semicommutative. Let $a = (2, 0)$ and $b = (0, 2)$ we have $ab = 0$, but $a\sigma(b) = (2, 0)(2, 0) = (4, 0) \neq 0$. Thus R is not semicommutative. Therefore the condition “ R is σ semicommutative” is not a superfluous condition in Proposition 3.9.

Proposition 2.6. Let σ be an endomorphism of a ring R . If $R[x; \sigma]$ is a semicommutative ring then R is σ -skew McCoy.

Proof. Let $f(x) \in R[x; \sigma]$ and consider the right ideal $I = f(x)R[x; \sigma]$. We claim that $r_{R[x; \sigma]}(f(x)R[x; \sigma])$ is σ -stable. Let $\varphi(x) \in R[x; \sigma]$ and $h(x) \in r_{R[x; \sigma]}(f(x)R[x; \sigma])$. Then $f(x)\varphi(x)h(x) = 0$, since $R[x; \sigma]$ is semicommutative, we get $f(x)\varphi(x)xh(x) = 0$ which gives $f(x)\varphi(x)\sigma(h(x))x = 0$. Therefore $r_{R[x; \sigma]}(f(x)R[x; \sigma])$ is σ -stable. On other hand, let $g(x) \in R[x; \sigma] \setminus \{0\}$ such that $f(x)g(x) = 0$. We have $f(x)R[x; \sigma]g(x) = 0$, then there exists a nonzero $c \in R$ satisfying $f(x)R[x; \sigma]c = 0$ by Theorem 2.1. In particular, we have $f(x)c = 0$. Thus R is σ -skew McCoy. □

We have immediately the next Corollaries.

Corollary 2.7 ([5, Corollary 6]). Let σ be an endomorphism of a ring R . If $R[x; \sigma]$ is reversible then R is σ -skew McCoy.

Corollary 2.8 ([6, Corollary 2.3]). Let σ be an endomorphism of a ring R . If $R[x]$ is semicommutative then R is right McCoy.

Definition 2.9. Let R be a ring, M_R an R -module and σ an endomorphism of R . For $m \in M_R$ and $a \in R$. We say that M_R satisfies the condition (\mathcal{C}_σ^1) (resp., (\mathcal{C}_σ^2)) if $ma = 0$ (resp., $m\sigma(a)a = 0$) implies $m\sigma(a) = 0$.

Corollary 2.10. Let σ be an endomorphism of a ring R .

- (1) If R is semicommutative satisfying the condition (\mathcal{C}_σ^2) then it is σ -skew McCoy.
- (2) If R is reduced and right σ -reversible then it is σ -skew McCoy.

Proof. (1) Immediately from [20, Proposition 3.4]. (2) Clearly from (1). □

There is an example of a ring R and an endomorphism σ of R , such that $R[x; \sigma]$ is semicommutative and R is σ -skew McCoy.

Example 2.11. Consider the ring $R = \mathbb{Z}_2[x]$ where \mathbb{Z}_2 is the ring of integers modulo 2 and σ an endomorphism of R defined by $\sigma(f(x)) = f(0)$.

- (1) By [8, Example 5], R is σ -skew Armendariz then it's σ -skew McCoy.
- (2) R is a commutative domain so it's Baer then right p.q.-Baer ([9, Example 8]).
- (3) Since R is semicommutative and right p.q.-Baer, if we show that R satisfies (\mathcal{C}_σ^1) then by [20, Corollary 3.5], $R[y, \sigma]$ is semicommutative. Let $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_mx^m \in R$. Suppose that $fg = 0$ then we have the system of equations:

$$a_0b_0 = 0 \tag{0}$$

$$a_0b_1 + a_1b_0 = 0 \tag{1}$$

$$a_0b_2 + a_1b_1 + a_2b_0 = 0 \tag{2}$$

$$a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = 0 \tag{3}$$

$$\vdots$$

$$a_nb_m = 0 \tag{(n+m)}$$

Eq.(0) implies $a_0 = 0$ or $b_0 = 0$, if $b_0 = 0$ then $f\sigma(g) = fb_0 = 0$. We can suppose $b_0 \neq 0$. Then $a_0 = 0$. Eq.(1) implies $a_1b_0 = 0$. Eq.(2) implies $a_1b_1 + a_2b_0 = 0$, multiplying on the right by b_0 , we find $a_1b_1b_0 + a_2b_0^2 = 0$ then $a_2b_0^2 = 0$ because $a_1b_1b_0 = 0$ (since \mathbb{Z}_2 is semicommutative), also \mathbb{Z}_2 is reduced then $a_2b_0 = 0$. We continue with the same manner yields $a_ib_0 = 0$ for $i = 0, 1, \dots, n$. Thus $fb_0 = f\sigma(g) = 0$. Therefore R satisfies the condition (\mathcal{C}_σ^1) .

Example 2.12. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let $\sigma: R \rightarrow R$ be defined by $\sigma(a, b) = (b, a)$. Consider $p(x) = (1, 0) + (1, 0)x$ and $q(x) = (0, 1) + (1, 0)x \in R[x; \sigma]$, we have $p(x)q(x) = 0$. But $p(x)c \neq 0$ for any nonzero $c \in R$. Thus R is not σ -skew McCoy. Also $p(x)(1, 0)q(x) = (1, 0)x \neq 0$. Therefore $R[x; \sigma]$ is not semicommutative.

Example 2.13. Let \mathbb{Z}_3 be the ring of integers modulo 3. Consider the 2×2 matrix ring $R = \text{Mat}_2(\mathbb{Z}_3)$ over \mathbb{Z}_3 and an endomorphism $\sigma: R \rightarrow R$ be defined by

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

- (1) By [5, Example 11], R is not σ -skew McCoy.

(2) $R[x; \sigma]$ is not semicommutative. For

$$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x, \quad q(x) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x \in R[x; \sigma].$$

We have $p(x)q(x) = 0$, but $p(x) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} q(x) \neq 0$.

3 Nagata Extension and McCoyness

The next construction is due to Nagata [17]. Let R be a commutative ring, M_R be an R -module and σ an endomorphism of R . The R -module $R \oplus_\sigma M_R$ acquires a ring structure (possibly noncommutative), where the product is defined by $(a, m)(b, n) = (ab, n\sigma(a) + mb)$, where $a, b \in R$ and $m, n \in M_R$. We shall call this extension the *Nagata extension* of R by M_R and σ . If $\sigma = id_R$, then $R \oplus_{id_R} M_R$ (denoted by $R \oplus M_R$) is a commutative ring. Anderson and Camillo [1], have proved that if R is commutative domain then M_R is Armendariz if and only if $R \oplus M_R$ is Armendariz. We'll see that this result still true for $R \oplus_\sigma M_R$. Kim et al. [19], have proved that, if R is a commutative domain and σ is a monomorphism of R then $R \oplus_\sigma R$ is reversible, and so it is McCoy. Recall that if σ is an endomorphism of a ring R , then the map $R[x] \rightarrow R[x]$ defined by $\sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n \sigma(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends σ . We shall also denote the extended map $R[x] \rightarrow R[x]$ by σ and the image of $f \in R[x]$ by $\sigma(f)$. In this section, we'll discuss when the Nagata extension $R \oplus_\sigma M_R$ is McCoy. Let R be a commutative domain. The set $T(M) = \{m \in M | r_R(m) \neq 0\}$ is called the *torsion submodule* of M_R . If $T(M) = M$ (resp., $T(M) = 0$) then M_R is *torsion* (resp., *torsion-free*).

Proposition 3.1. *Let R be a commutative domain and M_R an R -module. Then $R \oplus_\sigma M_R$ is Armendariz if and only if M_R is Armendariz. In particular, if M_R is torsion-free then $R \oplus_\sigma M_R$ is Armendariz.*

Proof. Let $R' = R \oplus_\sigma M_R$, we have $R'[x] = R[x] \oplus_\sigma M[x]$. Suppose that R' is Armendariz. Let $m = \sum_{i=0}^p m_i x^i \in M[x]$ and $f = \sum_{j=0}^q a_j x^j \in R[x]$ with $mf = 0$. We have $(0, m) = \sum_{i=0}^p (0, m_i) x^i \in R'[x]$ and $(f, 0) = \sum_{j=0}^q (a_j, 0) x^j \in R'[x]$, since R' is Armendariz then $(0, m_i)(a_j, 0) = (0, m_i a_j) = (0, 0)$ for all i, j . Thus $m_i a_j = 0$ for all i, j . Conversely, suppose that M_R is Armendariz. Let $f, g \in R[x]$ and $m, n \in M[x]$ such that $(f, m)(g, n) = (0, 0)$. Write $(f, m) = \sum (a_i, m_i) x^i \in R'[x]$ and $(g, n) = \sum (b_j, n_j) x^j \in R'[x]$. From $(f, m)(g, n) = (0, 0)$, we have $(fg, n\sigma(f) + mg) = (0, 0)$. Since $R[x]$ is

a commutative domain, then $f = 0$ or $g = 0$. If $f = 0$, we get $mg = 0$. Then $m_i b_j = 0$ and $a_i = 0$ for all i, j . Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j \sigma(a_i) + m_i b_j) = (0, 0)$. Otherwise, we get $n\sigma(f) = 0$. Then $b_j = 0$ and $n_j \sigma(a_i) = 0$ for all i, j . Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j \sigma(a_i) + m_i b_j) = (0, 0)$. Therefore R' is Armendariz. \square

Corollary 3.2. *Let R be a commutative domain and M_R an R -module satisfying the condition $(\mathcal{C}_{id_R}^2)$. Then $R \oplus_\sigma M_R$ is Armendariz.*

Proof. Since M_R is semicommutative then it is Armendariz by [20, Lemma 3.3]. \square

Proposition 3.3. *Let R be a commutative ring and M_R an R -module such that R satisfies (\mathcal{C}_σ^1) and M_R satisfies (\mathcal{C}_σ^2) . Then $R \oplus_\sigma M_R$ is a semicommutative ring.*

Proof. We'll use freely the conditions (\mathcal{C}_σ^1) and (\mathcal{C}_σ^2) . Let $(r, m), (s, n) \in R \oplus_\sigma M_R$ such that

$$(r, m)(s, n) = (rs, n\sigma(r) + ms) = (0, 0). \quad (1)$$

We'll show that for any $(t, u) \in R \oplus_\sigma M_R$

$$(r, m)(t, u)(s, n) = (rts, n\sigma(rt) + u\sigma(r)s + mts) = (0, 0). \quad (2)$$

It suffices to show $n\sigma(rt) + u\sigma(r)s + mts = 0$. Multiplying $n\sigma(r) + ms = 0$ of Eq.(1) on the right hand by r , gives $n\sigma(r)r = 0$, so we get $n\sigma(r) = 0$ and hence $ms = 0$. Thus $n\sigma(rt) = mts = 0$. Clearly $rs = 0$ implies $\sigma(r)s = 0$ and so $u\sigma(r)s = 0$. Therefore $n\sigma(rt) + u\sigma(r)s + mts = 0$. \square

Proposition 3.4. *Let R be commutative domain and M_R an R -module. Then $R \oplus_\sigma M_R$ is a semicommutative right McCoy ring.*

Proof. Consider equations (1) and (2) of Proposition 3.3. From Eq.(1), we get $r = 0$ or $s = 0$ since R is a domain. Say $r = 0$, then $rts = n\sigma(rt) = u\sigma(r)s = 0$, and $mts = 0$ from (1), hence we have (2). Next say $s = 0$, it follows $rts = u\sigma(r)s = mts = 0$ and $n\sigma(rt) = 0$ from (1), and so we have (2). Therefore $(r, m)(R \oplus_\sigma M)(s, n) = 0$. For McCoyness, let $(r, m), (s, n) \in R' = R \oplus_\sigma M_R$. Suppose that $(r, m)(s, n)^2 = (rs^2, n\sigma(r^2) + ns\sigma(r) + ms^2) = 0$, then $r = 0$ or $s = 0$ which implies $(r, m)(s, n) = (rs, n\sigma(r) + ms) = 0$. Thus by Corollary 2.10(1), $R \oplus_\sigma M_R$ is right McCoy. \square

The next example shows that under the conditions of Proposition 3.4, $R \oplus_\sigma M_R$ can't be reversible.

Example 3.5. Let D be a commutative domain and $R = D[x]$ be the polynomial ring over D with an indeterminate x . Consider the endomorphism $\sigma: R \rightarrow R$ defined by $\sigma(f(x)) = f(0)$. Since $(x, 1)(0, 1) = (0, 0)$ and $(0, 1)(x, 1) = (0, x) \neq (0, 0)$, then $R \oplus_\sigma R$ is not reversible. Thus $R \oplus_\sigma M_R$ can't be reversible under the conditions of Proposition 3.4.

Lemma 3.6. If M_R is an Armendariz module. Let $m(x) \in M[x]$ and $f(x), g(x) \in R[x]$ such that $m(x) = \sum_{i=0}^n m_i x^i$, $f(x) = \sum_{j=0}^p a_j x^j$ and $g(x) = \sum_{k=0}^q b_k x^k$. Then

$$m(x)f(x)g(x) = 0 \Leftrightarrow m_i a_j b_k = 0 \text{ for all } i, j, k.$$

Proof. (\Leftarrow) Clear. (\Rightarrow) If $m(x)f(x) = 0$ then $m(x)a_j = 0$ for all j . Now, if $m(x)f(x)g(x) = 0$ then $m(x)[f(x)b_k] = 0$ for all k . Since M_R is Armendariz we have $m_i(a_j b_k) = 0$ for all i, j, k . Thus $m_i a_j b_k = 0$ for all i, j, k . \square

Lemma 3.7. If M_R is an Armendariz module satisfying the condition (\mathcal{C}_σ^2) . Then $M[x]_{R[x]}$ satisfies the condition (\mathcal{C}_σ^2) .

Proof. Let $m(x) = \sum_{i=0}^n m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^p a_j x^j \in R[x]$. Suppose that $m(x)\sigma(f(x))f(x) = 0$. By Lemma 3.6, $m_i \sigma(a_j) a_k = 0$ for all i, j, k . In particular $m_i \sigma(a_j) a_j = 0$ for all i, j . Then $m_i \sigma(a_j) = 0$ for all i, j . Therefore $m(x)\sigma(f(x)) = 0$. \square

Theorem 3.8. Let R be a commutative Armendariz ring, σ an endomorphism of R and M_R a module satisfying the condition (\mathcal{C}_σ^2) . Then M_R is Armendariz if and only if $R \oplus_\sigma M_R$ is Armendariz.

Proof. Let $f, g \in R[x]$ and $m, n \in M[x]$ such that $(f, m)(g, n) = (0, 0)$. Write $(f, m) = \sum (a_i, m_i) x^i \in R'[x]$ and $(g, n) = \sum (b_j, n_j) x^j \in R'[x]$. From $(f, m)(g, n) = (0, 0)$, we have $(fg, n\sigma(f) + mg) = (0, 0)$. Since R is Armendariz, then $a_i b_j = 0$ for all i, j . Multiplying $n\sigma(f) + mg = 0$ on the right by f , we have $n\sigma(f)f = 0$ by Lemma 3.7, we get $n\sigma(f) = 0$ and so $mg = 0$. Since M_R is Armendariz we have $m_i b_j = 0$ and $n_i \sigma(a_j) = 0$ for all i, j . Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j \sigma(a_i) + m_i b_j) = (0, 0)$. Therefore R' is Armendariz. The converse is clear. \square

Corollary 3.9. If R is a commutative reduced ring which satisfies the condition (\mathcal{C}_σ^1) then $R \oplus_\sigma R$ is semicommutative and Armendariz.

Proof. Immediately by Proposition 3.3 and Theorem 3.8. \square

Example 3.10. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let $\sigma: R \rightarrow R$ be defined by $\sigma(a, b) = (b, a)$. Clearly R is a commutative reduced ring but not a domain. Let $A = ((0, 1), (0, 1))$, $B = ((1, 0), (0, 1))$ and $C = ((1, 0), (1, 0))$. We have

$$AB = ((0, 1), (0, 1))((1, 0), (0, 1)) = ((0, 0), ((0, 1)\sigma(0, 1) + (0, 1)(1, 0))) = 0.$$

But

$$\begin{aligned} ACB &= ((0, 1), (0, 1))((1, 0), (1, 0))((1, 0), (0, 1)) = ((0, 0), (1, 0))((1, 0), (0, 1)) \\ &= ((0, 0), (1, 0)) \neq 0. \end{aligned}$$

Hence $R \oplus_{\sigma} R$ is not semicommutative. On other hand, we have $(1, 0)(0, 1) = 0$, but $(1, 0)\sigma((0, 1)) = (1, 0)(1, 0) = (1, 0) \neq 0$, so R does not satisfying the condition (C_{σ}^1) . Thus the condition (C_{σ}^1) in Corollary 3.9 is not superfluous.

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References

- [1] D.D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra 26(7), (1998), 2265-2272.
- [2] S. Annin, *Associated primes over skew polynomials rings*, Comm. Algebra 30 (2002), 2511-2528.
- [3] M. Baser, A. Harmanci and T.K. Kwak, *Generalized semicommutative rings and their extensions*, Bull. Korean Math. Soc. 45(2) (2008), 285-297.
- [4] W.E. Clark, *Twisted matrix units semigroup algebras*, *Duke Math.Soc.*, **35** (1967), 417-424.
- [5] M. Baser, T.K. Kwak and Y. Lee, *The McCoy condition on skew polynomial rings*, Comm. Algebra 37(11) (2009), 4026-4037.

- [6] Y. Hirano, *On annihilator ideals of polynomial ring over a noncommutative ring*, J. Pure. Appl. Algebra 168 (1) (2002), 45-52.
- [7] C.Y. Hong, N.K. Kim and T.K. Kwak, Ore extensions of Baer and p.p.-rings, *J. Pure and Appl. Algebra*, **151(3)** (2000), 215-226.
- [8] C.Y. Hong, N.K. Kim and T.K. Kwak, *On Skew Armendariz Rings*, Comm. Algebra 31(1) (2003), 103-122.
- [9] C.Y. Hong, T.K. Kwak and S.T. rezvi, *Extensions of generalized Armendariz rings*, Algebra Colloq. 13(2) (2006), 253-266.
- [10] C.Y. Hong, N.K. Kim and Y. Lee, *Ore Extensions of Quasi-Baer Rings*, Comm. Algebra 37(6) (2009), 2030-2039.
- [11] C.Y. Hong, N.K. Kim, T.K. Kwak and Y. Lee, *Extensions of McCoy's Theorem*, Galas. Math. J. 52 (2010) 155-159.
- [12] C.Y. Hong, Y.C. Jeon, N.K. Kim and Y. Lee, *The McCoy condition on noncommutative rings*, Comm. Algebra 39(5) (2011), 1809-1825.
- [13] C. Huh, Y. Lee and A. Smoktunowics, *Armendariz rings and semicommutative rings*, Comm. Algebra 30 (2) (2002), 751-761
- [14] C. Huh, H.K. Kim, N.K. Kim and Y. Lee, *Basic examples and extensions of symmetric rings*, J. Pure and Appl. Algebra 202 (2005), 154-167.
- [15] N.H. McCoy, *Remarks on divisors of zero*, Amer. math. monthly 49 (1942), 286-295.
- [16] N.H. McCoy, *Annihilators in polynomial rings*, Amer. math. monthly 64 (1957), 28-29.
- [17] M. Nagata, *Local rings*, Interscience, New York, 1962.
- [18] P.P. Nielson, *Semicommutative and McCoy condition*, J. Pure Appl. Algebra 298 (2006), 134-141.
- [19] N.K. Kim, Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra 185 (2003), 207-223.

- [20] M. Louzari, *On skew polynomials over $p.q$ -Baer and $p.p$ -modules*, Inter. Math. Forum 6 (2011), no. 35, 1739 - 1747.
- [21] C.P. Zhang and J.L. Chen, *σ -skew Armendariz modules and σ -semicommutative modules*, Taiwanese J. Math. 12(2) (2008), 473-486.